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Second Order Corrections to the Muon Lifetime and the Semileptonic B Decay

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Abstract

In this article two-loop QED corrections to the muon decay and corrections of order α_s^2 to the semileptonic decay of the bottom quark are considered. We compute the imaginary part of the four-loop diagrams contributing to the corresponding fermion propagator in the limit of small external momentum. The on-shell condition is obtained with the help of a conformal mapping and Padé approximation. Via this method we confirm the existing results by an independent calculation.

1 Introduction

The Fermi coupling constant, G_F, constitutes together with the electromagnetic coupling constant and the mass of the Z boson the most precise input parameters of the Standard Model of elementary particle physics. G_F is defined through the muon lifetime, and the decay of the muon, as a purely leptonic process, is rather clean — both experimentally and theoretically. The one-loop corrections of order α were computed more than 40 years ago [1] whereas only recently the two-loop corrections of order α^2 have been evaluated [2]. The large gap in time shows that this calculation is highly non-trivial. The inclusion of the two-loop terms removed the relative theoretical error of 1.5×10^{-5} which was an estimate on the size of the missing corrections. The remaining error on G_F now reads 0.9×10^{-5} and is of pure experimental nature. Upcoming experiments will further improve the accuracy of the muon lifetime measurement and therefore the $\mathcal{O}(\alpha^2)$ corrections to the muon decay are very important and constitute a crucial ingredient from the theoretical side. These facts make it desirable to have an independent check on the correctness of the $\mathcal{O}(\alpha^2)$ result. We

also want to mention that in a recent article [3] optimization methods have been used in order to estimate the coefficient of order α^3.

In view of the upcoming B physics experiments the evaluation of quantum corrections to B meson properties have become topical. In particular it is possible to use the semileptonic decay rate of the bottom quark in order to determine the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements with quite some accuracy. In [4] an approximate expression for the $\mathcal{O}(\alpha_s^2)$ corrections of $\Gamma(b \to cl\bar{\nu}_l)$ has been obtained where a non-vanishing charm quark mass has been included. In the decay $b \to ue\bar{\nu}_e$ the mass of the u quark can be neglected which reduces the calculation to the situation given in the muon decay. The only additional diagrams are those which arise due to the non-abelian structure of QCD. The results of order α_s^2 have been obtained in [5]. The corrections proportional to the number of light quarks have already been computed in [6].

In this letter we confirm through an independent calculation the results of order α^2 to the muon decay [2] and the one of order α_s^2 to the semileptonic bottom quark decay [5]. In the next section our method is discussed, in Section 3 the results are presented.

2 Method and Notation

In [2, 5] the imaginary part of the muon propagator is computed up to the four-loop level. Recurrence relations based on the integration-by-parts technique [7] are used in order to reduce the integrals to be evaluated to a minimal set — so-called master integrals. They are finally evaluated by computing expansions in the ratio of external momentum and internal muon mass. It is possible to take the on-shell limit and perform the infinite sum which finally leads to an exact result for the integrals. For concise reviews of expansion methods see e.g. [8, 9].

In contrast to [2, 5] our approach is based on an expansion of the full fermion propagator in the limit

$$M^2 \gg q^2 \,, \tag{1}$$

where q is the external momentum and M is the propagator mass of the muon and bottom quark, respectively[1]. Throughout the whole paper we will neglect effects induced by the non-vanishing electron and up quark mass. The on-shell limit $q^2 \to M^2$ will be performed afterwards with the help of Padé approximations. This, of course, only provides an approximation to the exact result. However, the integrals to be evaluated are simplified considerably. We will demonstrate that the accuracy obtained with our method is sufficient to check the existing results and enables the same reduction of the theoretical error on G_F.

The notation is essentially adopted from [10], where corrections of $\mathcal{O}(\alpha_s^2)$ to the decay $t \to Wb$ have been computed. For completeness we briefly repeat in this paper the main formulae. The decay rate — both for the muon and the bottom quark — can be written

[1] There are, of course, diagrams that do not contain an internal propagator of mass M. These diagrams are computed without any expansion.

in the form

$$\Gamma \;=\; 2M\,\mathrm{Im}\left[z\,S_V^{OS} - S_S^{OS}\right]\Big|_{z=1}\,, \tag{2}$$

where

$$S_S^{OS} \;=\; Z_2^{OS} Z_m^{OS}\left(1-\Sigma_S^0\right)\,, \qquad\qquad S_V^{OS} \;=\; Z_2^{OS}\left(1+\Sigma_V^0\right)\,, \tag{3}$$

are functions of the variable

$$z = \frac{q^2}{M^2}\,. \tag{4}$$

M is the on-shell mass. Σ_S^0 and Σ_V^0 represent the scalar and vector part of the corresponding fermion propagator, respectively. They are functions of the external momentum q and the bare mass m^0 of the fermion under consideration. In our case they further depend on the bare electromagnetic coupling α^0 and the strong coupling constant α_s^0, respectively, and are proportional to the square of the Fermi coupling constant, G_F^2.

The mass renormalization constant, Z_m^{OS}, entering in (3) can be extracted from [11]. In contrast, Z_2^{OS} has to be evaluated in the limit $M^2 \gg q^2$ since the handling of Z_2^{OS} is determined by the computation of the fermion propagator. As we are only interested in the imaginary part and furthermore consider only QED or QCD corrections to the leading order term the quantity Z_2^{OS} has to be known up to two loops only. The result can be taken from [10]. The renormalization of α and α_s proceeds in the usual way where we have chosen to renormalize also the electromagnetic coupling in a first step in the $\overline{\mathrm{MS}}$ scheme.

In order to get reliable results it is necessary to compute as many terms as possible in the expansion parameter z. Subsequently a Padé approximation is applied which is at length described in [10]. We just want to mention that before the Padé procedure a conformal mapping can be used which maps the complex z-plane into the interiour of the unit circle. Following Ref. [10] we denote those results by ω-Padés and the ones obtained without conformal mapping by z-Padés.

Some Padé approximants develop poles inside the unit circle ($|z| \leq 1$ and $|\omega| \leq 1$, respectively) in conflict with the analycity of the exact result. In general we will discard such numbers in the following. In some cases, however, the pole coincides with a zero of the numerator up to several digits accuracy, and these Padé approximations will be included in our sample. To be precise: in addition to the Padé results without any poles inside the unit circle, we will use the ones where the poles are accompanied by zeros within a circle of radius 0.01, and the distance between the pole and the physically relevant point $q^2/M^2 = 1$ is larger than 0.1.

The central values and the estimated uncertainty will be extracted from Padé results $[m/n]$ with $m+n$ not too small and $|m-n| \leq 2$. The central value is obtained by averaging the Padé results and the uncertainty is given by the maximal deviation from the central value.

It is convenient to parameterize the radiative corrections for the semileptonic bottom quark decay in the following form:

$$\Gamma(b \to u e \bar{\nu}_e) \;=\; \Gamma_b^0\left[A_b^{(0)} + \frac{\alpha_s}{\pi} C_F A_b^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 A_b^{(2)} + \ldots\right]\,,$$

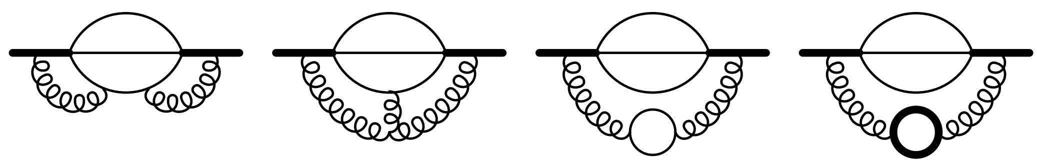

Figure 1: Sample diagrams for the fermion self energy. In the case of the bottom quark (thick line) decay two of the thin lines represent the lepton pair and the third one the up quark. All (one-particle-irreducible) diagrams involving the coupling of the gluons (loopy lines) to the up and bottom quark have to be taken into account. The modifications in the case of the muon decay are obvious.

$$A_b^{(2)} = C_F^2 A_{b,A}^{(2)} + C_A C_F A_{b,NA}^{(2)} + C_F T n_l A_{b,l}^{(2)} + C_F T A_{b,F}^{(2)} \,, \tag{5}$$

with $\Gamma_b^0 = G_F^2 M_b^5 |V_{ub}|^2/(192\pi^3)$. For QCD the colour factors are given by $C_F = 4/3$, $C_A = 3$, and $T = 1/2$. n_l is the number of massless quark flavours and will be set to $n_l = 4$ at the end. $A_{b,A}^{(2)}$ corresponds to the abelian part already present in QED, $A_{b,NA}^{(2)}$ represents the non-abelian contribution, and $A_{b,l}^{(2)}$ and $A_{b,F}^{(2)}$ denote the corrections involving a second fermion loop with massless and heavy quarks, respectively. In principle there is also a contribution involving a virtual top quark loop. It is, however, suppressed by M_b^2/M_t^2 and will thus be neglected. In Fig. 1 a representative diagram for each one of these four colour structures is pictured. In Eq. (5) $\alpha_s = \alpha_s(\mu)$ is defined with five active flavours. The analytic result for $A_b^{(2)}$ can be found in [5].

An analogous expression to (5) can also be defined in the case of the muon decay

$$\begin{aligned} \Gamma(\mu \to \nu_\mu e \bar{\nu}_e) &= \Gamma_\mu^0 \left[A_\mu^{(0)} + \frac{\bar{\alpha}}{\pi} A_\mu^{(1)} + \left(\frac{\bar{\alpha}}{\pi}\right)^2 A_\mu^{(2)} + \ldots \right] , \\ A_\mu^{(2)} &= A_{\mu,\gamma\gamma}^{(2)} + A_{\mu,e}^{(2)} + A_{\mu,\mu}^{(2)} \,, \end{aligned} \tag{6}$$

with $\Gamma_\mu^0 = G_F^2 M_\mu^5/(192\pi^3)$. $\bar{\alpha} = \bar{\alpha}(\mu)$ represents the electromagnetic coupling in the $\overline{\mathrm{MS}}$ scheme. $A_{\mu,\gamma\gamma}^{(2)}$ represents the purely photonic corrections whereas $A_{\mu,e}^{(2)}$ and $A_{\mu,\mu}^{(2)}$ contain an additional electron and muon loop, respectively. The contribution involving a virtual τ loop is not listed in (6) as it is suppressed by M_μ^2/M_τ^2 and almost four orders of magnitudes smaller than the other terms [12].

Let us in the following describe our method used for the practical calculation in the case of the μ decay. The difference to the quark decay consists only in the transition from QED to QCD which increases the number of diagrams and makes it necessary to include the colour factors; the idea is, however, applicable in the very same way.

Following common practice, we investigate the effective theory where the W boson is integrated out. The QED corrections to the resulting Fermi contact interaction were shown to be finite to all orders [13]. It is quite advantageous to perform a Fierz transformation which for a pure $V - A$ theory has the consequence that afterwards the two neutrino

lines appear in the same fermion trace. Thus the QED corrections only affect the fermion trace involving the muon and the electron. This also provides some simplifications in the treatment to γ_5 since in the case of vanishing electron mass a fully anticommuting prescription can be used.

As described above we consider the fermion two-point functions and compute the imaginary part arising from the intermediate states with two neutrinos and the electron. As a consequence already for the Born result a two-loop diagram has to be considered. However, it turns out that the loop integration connected to the two neutrino lines can be performed immediately as it constitutes a massless two-point function. This is also the case after allowing for additional photonic corrections. As a result one encounters in the resulting diagram a propagator with one of the momenta raised to power ε where $D = 4 - 2\varepsilon$ is the space-time dimension. This slightly increases the difficulty of the computation of the resulting diagrams. Especially for the order α^2 corrections, where the original four-loop diagrams are reduced to three-loop ones with non-integer powers of denominators, it is a priori not clear that these integrals can be solved analytically. However, it turns out that for the topologies needed in our case this is indeed possible. For the computation of the massless two-point functions we have used the package `MINCER` [14]. Slight modifications enabled us to use this package also for the computation of the new type of integrals.

The calculation is performed with the help of the package `GEFICOM` [15]. It uses `QGRAF` [16] for the generation of the diagrams and `EXP` [17] for the application of the asymptotic expansion procedures. For more technical details we refer to a recent review concerned with the automatic computation of Feynman diagrams [18].

The dispersive and absorbtive part of the fermion self energies are gauge dependent for $q^2 \neq M^2$. Hence the dependence on the gauge parameter ξ in Eq. (2) only drops out after summing infinitely many terms in the expansion around $z = 0$ and setting $z = 1$. Since we are only dealing with a limited number of terms, our approximate results will still depend on the choice of ξ even after taking $z \to 1$. It is clear that with extreme values of ξ almost any number could be produced. Thus the question arises which value of ξ should be taken in order to arrive at a reliable prediction for the decay rates. As the one-loop corrections can be evaluated for an arbitrary gauge parameter an extensive study can be performed and we can gain some hints for the choice of ξ at order α^2.

3 Results

Let us in a first step present the results for the muon decay and afterwards discuss the additional diagrams necessary for the QCD corrections to the bottom quark decay.

The lowest order (Born) diagram can be computed directly. In this case the electron mass can be chosen different from zero and an expansion in M_e^2/M_μ^2 can be performed reproducing the exact result

$$A_\mu^{(0)} \;=\; 1 - 8\,\frac{M_e^2}{M_\mu^2} - 12\,\frac{M_e^4}{M_\mu^4}\,\ln\frac{M_e^2}{M_\mu^2} + 8\,\frac{M_e^6}{M_\mu^6} - \frac{M_e^8}{M_\mu^8}\,. \tag{7}$$

input	P.A.	$\xi=-2$	$\xi=-1$	$\xi=-1/2$	$\xi=0$	$\xi=1/2$	$\xi=1$	$\xi=2$
7	[4/2]	−1.899	−1.857	−1.821	−1.792	−1.763	−1.735	−1.679
7	[3/3]	−1.894	−1.854	−1.821	−1.791	−1.769	−1.741	−1.688
7	[2/4]	−1.900	−1.858	−1.822	−1.792	−1.765	−1.736	−1.673
8	[4/3]	−1.880	—	−1.821	−1.804	−1.777	−1.755	−1.712
8	[3/4]	−1.881	—	−1.821	−1.805	−1.777	−1.753	−1.711
9	[5/3]	−1.870	—	−1.821	−1.802	−1.783	−1.764	−1.728
9	[4/4]	−1.868	—	$-1.821^{(\star)}$	−1.802	−1.785	−1.767	−1.732
9	[3/5]	−1.871	—	−1.821	−1.802	−1.783	−1.763	−1.726
exact:		−1.810						

Table 1: z-Padé results for the $\mathcal{O}(\alpha)$ corrections for different choices of ξ.

We want to demonstrate the power of our method for the order α corrections where four three-loop diagrams contribute. At this order we are able to evaluate a large number of moments which gives us a suggestion how many terms are necessary at $\mathcal{O}(\alpha^2)$ in order to obtain reliable results. Furthermore the computation can be performed for arbitrary gauge parameter which also provides some hints for the two-loop QED corrections.

Applying the asymptotic expansion in the limit $M_\mu^2 \gg q^2$ to the four three-loop diagrams contributing to the $\mathcal{O}(\alpha)$ correction leads to the following result for the first nine expansion terms

$$
\begin{aligned}
A^{(1)}_{\mu,exp} = & -\frac{11}{8} + \frac{25}{48}\,\xi + z_\mu \left(-\frac{61}{400} - \frac{1}{5}\,\xi\right) + z_\mu^2 \left(-\frac{47}{540} - \frac{1}{12}\,\xi\right) \\
& + z_\mu^3 \left(-\frac{6929}{132300} - \frac{1}{21}\,\xi\right) + z_\mu^4 \left(-\frac{11923}{352800} - \frac{1}{32}\,\xi\right) + z_\mu^5 \left(-\frac{439213}{19051200} - \frac{1}{45}\,\xi\right) \\
& + z_\mu^6 \left(-\frac{156487}{9525600} - \frac{1}{60}\,\xi\right) + z_\mu^7 \left(-\frac{931367}{76839840} - \frac{1}{77}\,\xi\right) \\
& + z_\mu^8 \left(-\frac{216409}{23522400} - \frac{1}{96}\,\xi\right) + \mathcal{O}(z_\mu^9)\,. \qquad (8)
\end{aligned}
$$

In Tab. 1 results for the $\mathcal{O}(\alpha)$ coefficient can be found where the Padé approximation is performed in the variable $z_\mu = q^2/M_\mu^2$. Furthermore the gauge parameter defined through the photon propagator $i(-g^{\mu\nu}+\xi q^\mu q^\nu/q^2)/(q^2+i\epsilon)$ is varied between[2] $\xi=-2$ and $\xi=+2$. Padé results which develop poles for $|z_\mu| \le 1$ are in general represented by a dash. If an approximate cancellation with a zero from the numerator takes place (see the discussion above), they are marked by a star ($\star$). The comparison with the exact result shows that for all values of ξ reasonable agreement is found. However, there is a clear preference for $\xi=0$ where the agreement with the exact result is best[3]. Thus we will adopt this value

[2]Despite the fact that for $\xi > 1$ the generating functional is in principle not defined we decided to choose this range for the gauge parameter.

[3]It seems that the ξ-dependent terms of $A^{(1)}_{\mu,exp}$ follow from the construction rule $25/48 - \sum_{n=1}^{\infty} z_\mu/n/(n+4)$ which for $z_\mu = 1$ indeed gives zero.

input	P.A.	z	ω
7	[4/2]	-1.792	$-1.790^{(\star)}$
7	[3/3]	-1.791	-1.811
7	[2/4]	-1.792	-1.816
8	[4/3]	-1.804	-1.803
8	[3/4]	-1.805	-1.788
9	[5/3]	-1.802	-1.807
9	[4/4]	-1.802	—
9	[3/5]	-1.802	-1.808
exact:		-1.810	

Table 2: Padé results for the $\mathcal{O}(\alpha)$ corrections (z- and ω-Padés).

for the $\mathcal{O}(\alpha^2)$ calculation.

In Tab. 2 the gauge parameter is fixed to $\xi = 0$ and in addition to the z-Padés also the ω-Padés are listed. With the inclusion of more moments the approximation to the exact result improves. Taking only those results into account where eight or nine input terms enter the following result for the order α correction can be deduced

$$A^{(1)} = -1.80(1)\,. \tag{9}$$

Here the notation $1.80(1) = 1.80 \pm 0.01$ has been adopted. The excellent agreement with the exact result quoted in Tab. 2 encourages the use of our method.

At order α^2 only the first eight moments are at hand. Using only seven and eight input terms changes the numbers of Eq. (9) to

$$A^{(1)} = -1.80(2)\,, \tag{10}$$

which is still sufficiently accurate.

Let us now move on to the two-loop QED corrections. Altogether 44 four-loop diagrams contribute. The application of the asymptotic expansion in the limit $M_\mu^2 \gg q^2$ leads to 72 sub- and cosub-diagrams, which have to be evaluated. The analytical expressions obtained from the asymptotic expansion are quite lengthy. Thus we refrain from listing the results explicitly and present them only in numerical form.

Our results at $\mathcal{O}(\alpha^2)$ are summarized in Tab. 3 where the scale $\mu^2 = M_\mu^2$ has been adopted[4]. For comparison in the last line the numbers presented in Ref. [2] are listed. The results one obtains using the Padé approximants computed with seven and eight input terms read

$$\begin{aligned} A^{(2)}_{\mu,\gamma\gamma} &= 3.5(4)\,, \\ A^{(2)}_{\mu,e} &= 3.2(6)\,, \\ A^{(2)}_{\mu,\mu} &= -0.0364(1)\,. \end{aligned} \tag{11}$$

[4]The $\ln \mu^2/M_\mu^2$ terms can be reconstructed with the help of the β function governing the running of $\bar{\alpha}(\mu)$.

		$\gamma\gamma$		e		μ	
input	P.A.	z	ω	z	ω	z	ω
6	[3/2]	3.028	3.876	2.882	3.237	—	—
6	[2/3]	3.029	3.862	2.893	3.397	$-0.0365^{(\star)}$	—
7	[4/2]	3.067	3.815	2.931	3.898	-0.0364	—
7	[3/3]	3.062	—	2.969	3.332	-0.0364	—
7	[2/4]	3.066	3.841	2.959	3.353	—	—
8	[4/3]	—	$3.919^{(\star)}$	3.000	$3.287^{(\star)}$	-0.0364	—
8	[3/4]	—	$3.869^{(\star)}$	3.002	$3.581^{(\star)}$	-0.0364	—
exact:		3.56		3.22		-0.0364	

Table 3: Padé results for $A^{(2)}_{\mu,\gamma\gamma}$, $A^{(2)}_{\mu,e}$ and $A^{(2)}_{\mu,\mu}$.

input	P.A.	z	ω
6	[3/2]	5.836	$7.249^{(\star)}$
6	[2/3]	5.836	7.057
7	[4/2]	5.935	7.040
7	[3/3]	$5.833^{(\star)}$	7.076
7	[2/4]	5.938	7.080
8	[4/3]	6.110	$6.873^{(\star)}$
8	[3/4]	6.113	$7.060^{(\star)}$
exact:		6.743	

Table 4: Padé results for the corrections of $\mathcal{O}(\alpha^2)$ to the muon decay, $A^{(2)}_{\mu}$.

It is remarkable that the central values agree well with the exact results which can be interpreted as a sign that the presented error estimations are quite conservative. Furthermore it can be claimed that via our method we were able to confirm the results of [2].

The error of $A^{(2)}_{\mu,\mu}$ is particularly small, as the expansion in z converges very quickly. Note that in this case all ω-Padés develop poles inside the unit circle. The z-Padés are, however, very stable. A similar behaviour has been found for the analogous contribution to the decay of the top quark into a W boson and a bottom quark [10].

A prediction for the decay rate of the muon up to order α^2 can be in principle obtained by summing the individual terms of (11). This would, however, significantly overestimate the error. It is more promising to add in a first step the moments of the single contributions and perform the Padé procedure for the sum. The corresponding results are shown in Tab. 4 which finally lead to

$$A^{(2)}_{\mu} = 6.5(7)\,. \tag{12}$$

The deviation of the central value from the exact result of 6.743 is less than 3% and well covered by the extracted error of roughly 10%. Thus the sole knowledge of our results

input	P.A.	z	ω
6	[3/2]	-8.374	-9.253
6	[2/3]	-8.376	-9.421
7	[4/2]	-8.469	-9.164
7	[3/3]	-8.560	-9.076
7	[2/4]	-8.476	-9.288
8	[4/3]	-8.616	-9.073
8	[3/4]	-8.616	-9.073
exact:		-9.046	

Table 5: Padé results for the non-abelian part of $\mathcal{O}(\alpha_s^2)$, $A^{(2)}_{b,NA}$.

would also reduce the theoretical error on G_F as mentioned in the Introduction. Using the results presented in this paper the decay rate of the muon reads

$$\Gamma(\mu \to \nu_\mu e \bar{\nu}_e) \;=\; \Gamma^0_\mu \left[0.9998 - 1.810\frac{\bar{\alpha}(M_\mu)}{\pi} + 6.5(7)\left(\frac{\bar{\alpha}(M_\mu)}{\pi}\right)^2 + \ldots\right] . \tag{13}$$

As already noted in [3] the numerical coefficient in front of the second order corrections becomes very small if one uses the on-shell scheme for the definition of the coupling constant α. Then the $\overline{\rm MS}$ coupling is given by $\bar{\alpha}(M_\mu) = \alpha(1 + \alpha/3\pi \ln(M_\mu^2/M_e^2))$ and there is an accidental cancellation between the constant and the logarithm in the second order corrections.

Let us now turn to the semileptonic decay of the bottom quark. The Born and one-loop corrections can, of course, be taken from the muon decay. In particular we have $A^{(0)}_\mu = A^{(0)}_b$ and $A^{(1)}_\mu = A^{(1)}_b$. As far as the two-loop terms are concerned only the non-abelian contribution, $A^{(2)}_{b,NA}$, has to be computed in addition. The other colour structures are related to the expressions occurring in the muon decay rate through $A^{(2)}_{b,A} = A^{(2)}_{\mu,\gamma\gamma}$, $A^{(2)}_{b,l} = A^{(2)}_{\mu,e}$, and $A^{(2)}_{b,F} = A^{(2)}_{\mu,\mu}$, with obvious replacements of the masses.

In Tab. 5 the results for $A^{(2)}_{b,NA}$ can be found. We infer

$$A^{(2)}_{b,NA} \;=\; -8.8(4)\,, \tag{14}$$

where $\mu^2 = M_b^2$ has been chosen. The central value is again in very good agreement with the exact result [5] and agrees within the error estimate of 5%.

In order to get predictions for $A^{(2)}_b$ we again add the moments in a first step and perform the Padé approximations afterwards. From the results listed in Tab. 6 we deduce

$$A^{(2)}_b \;=\; -21.1(6)\,. \tag{15}$$

This number is in good agreement with the one stated in [5]. The error is quite small and amounts only to roughly 3%. The semileptonic decay rate of the bottom quark finally

input	P.A.	z	ω
6	[3/2]	-20.587	-21.835
6	[2/3]	-20.592	-22.252
7	[4/2]	-20.744	-21.159
7	[3/3]	-20.836	—
7	[2/4]	-20.757	-21.649
8	[4/3]	-20.964	-21.213
8	[3/4]	-20.965	-21.608
exact:		-21.296	

Table 6: Padé results for the corrections of $\mathcal{O}(\alpha_s^2)$ to $b \to ue\nu_e$, $A_b^{(2)}$.

reads

$$\Gamma(b \to ue\bar{\nu}_e) \;=\; \Gamma_b^0 \left[1 - 2.413\frac{\alpha_s(M_b)}{\pi} - 21.1(6)\left(\frac{\alpha_s(M_b)}{\pi}\right)^2 + \ldots\right]. \tag{16}$$

4 Conclusions

In this paper the two-loop QED corrections to the decay rate of the muon have been evaluated. A new method has been used in order to confirm via an independent calculation the result of Ref. [2]. From the muon decay the Fermi coupling constant, G_F, is determined which constitutes one of the basic input parameters of the Standard Model. Thus it is very important to have independent checks on such highly non-trivial computations. The inclusion of the new correction terms removes the theoretical error on G_F. After the additional computation of the non-abelian diagrams the QCD corrections to the semileptonic decay rate of the bottom quark $\Gamma(b \to ue\nu_e)$ are obtained. Again agreement with the literature [5] is found.

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